# An action approach to nodal and least energy normalized solutions for NLS

## Séminaire MAC
## Institut de mathématiques de Toulouse

Damien Galant

CERAMATHS/DMATHS
Université Polytechnique Hauts-de-France

Département de Mathématique
Université de Mons
F.R.S.-FNRS Research Fellow

Joint work with Colette De Coster (CERAMATHS/DMATHS, Valenciennes, France), Simone Dovetta and Enrico Serra (Politecnico di Torino, Italy)

Wednesday 25 June 2025

# Acknowledgments

First of all, let me thank:

# Acknowledgments

First of all, let me thank:

- Boris Shakarov and Elio Durand-Simonnet for their invitation;

# Acknowledgments

First of all, let me thank:

- Boris Shakarov and Elio Durand-Simonnet for their invitation;
- the ANR "NQG" project, represented by Boris, Elio, Romain Duboscq and Stefan Le Coz in Toulouse;

# Acknowledgments

First of all, let me thank:

- Boris Shakarov and Elio Durand-Simonnet for their invitation;
- the ANR "NQG" project, represented by Boris, Elio, Romain Duboscq and Stefan Le Coz in Toulouse;
- the organizers of the MAC seminar, thanks to David Lafontaine for the organization of this talk;

# Acknowledgments

First of all, let me thank:

- Boris Shakarov and Elio Durand-Simonnet for their invitation;
- the ANR “NQG” project, represented by Boris, Elio, Romain Duboscq and Stefan Le Coz in Toulouse;
- the organizers of the MAC seminar, thanks to David Lafontaine for the organization of this talk;
- my coauthors Colette De Coster (CERAMATHS/DMATHS, Valenciennes, France), Simone Dovetta (Politecnico di Torino, Italy) and Enrico Serra (PoliTO);

# Acknowledgments

First of all, let me thank:

- Boris Shakarov and Elio Durand-Simonnet for their invitation;
- the ANR "NQG" project, represented by Boris, Elio, Romain Duboscq and Stefan Le Coz in Toulouse;
- the organizers of the MAC seminar, thanks to David Lafontaine for the organization of this talk;
- my coauthors Colette De Coster (CERAMATHS/DMATHS, Valenciennes, France), Simone Dovetta (Politecnico di Torino, Italy) and Enrico Serra (PoliTO);
- you!

# The paper

De Coster C., Dovetta S., Galant D., Serra E.
*An action approach to nodal and least energy normalized solutions for nonlinear Schrödinger equations.* ArXiV preprint:
https://arxiv.org/abs/2411.10317
Accepted for publication in Annales de l'Institut Henri Poincaré C - Analyse non linéaire (2025).

# The nonlinear Schrödinger *evolution* equation

We consider the problem

$$\begin{cases} \boldsymbol{i}\partial_t\psi = -\Delta\psi - |\psi|^{p-2}\psi, & (t,x) \in [0,T[\times\Omega, \\ \psi(t,x) = 0, & (t,x) \in [0,T[\times\partial\Omega, \\ \psi(0,x) = \psi_0(x), & \psi_0 :\to \mathbb{C}, x \in \Omega \end{cases} \qquad (\mathrm{NLS_{evol}})$$

where

- $\psi : [0,T[\times\Omega \to \mathbb{C}$, $\Omega$ *bounded* domain in $\mathbb{R}^N$, $N \geq 1$;
- $\boldsymbol{i}^2 = -1$;
- $\partial_t\psi$ is the derivative with respect to the time variable;
- $\Delta = \sum_{1\leq i\leq N} \partial_{x_i}^2$ is the Laplacian on $\Omega$;
- $p > 2$ is a real parameter.

# Conservation laws

At least formally, the $L^2$ *norm* (the *mass*)

$$\|\psi(t,\cdot)\|_{L^2}^2 := \int_\Omega |\psi(t,x)|^2 \,\mathrm{d}x$$

and the *energy*

$$E\Big(\psi(t,\cdot)\Big) := \frac{1}{2}\int_\Omega |\nabla_x \psi(t,x)|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\psi(t,x)|^p \,\mathrm{d}x$$

are preserved during the evolution.

# Solitary wave solutions

Opposed to blow-up: *solitary waves* of the form

$$\psi(t,x) = e^{i\lambda t} u(x)$$

where $u \in H_0^1(\Omega;\mathbb{R}) = H_0^1(\Omega)$ is a solution of

$$-\Delta u + \lambda u = |u|^{p-2} u. \tag{NLS}$$

Some vocabulary:

- $\lambda \in \mathbb{R}$ is the *frequency* of the solitary wave;
- $\|u\|_{L^2}^2 = \|\psi(t,\cdot)\|_{L^2}^2$ is its *mass*.

# Two problems

**Problem**

*Given* $\lambda \in \mathbb{R}$, *how to find a nonzero stationary wave of frequency* $\lambda$*?*

**Problem**

*Given* $\mu > 0$, *how to find a stationary wave of mass* $\mu$*?*

Vocabulary: solutions with a prescribed mass are usually called *normalized solutions*.

# Two functionals

We recall that the *energy functional* is given by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \, \mathrm{d}x.$$

# Two functionals

We recall that the *energy functional* is given by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x.$$

Given $\lambda \in \mathbb{R}$, we also define the *action functional* by

$$\begin{aligned} J_\lambda(u) &:= E(u) + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x. \end{aligned}$$

# Variational formulations

## Proposition

*Given* $2 < p < 2^*$ *and* $\lambda \in \mathbb{R}$, *solutions of frequency* $\lambda$ *correspond to critical points of* $J_\lambda$ *on* $H_0^1(\Omega)$.

# Variational formulations

## Proposition

*Given $2 < p < 2^*$ and $\lambda \in \mathbb{R}$, solutions of frequency $\lambda$ correspond to critical points of $J_\lambda$ on $H_0^1(\Omega)$.*

## Proposition

*Given $2 < p < 2^*$ and $\mu > 0$, normalized solutions of mass $\mu$ correspond to constrained critical points of $E$ on the $L^2$-sphere*

$$\mathcal{M}_\mu := \left\{ u \in H_0^1(\Omega) \mid \|u\|_{L^2(\Omega)} = \mu \right\}.$$

# Variational formulations

**Proposition**

*Given* $2 < p < 2^*$ *and* $\lambda \in \mathbb{R}$, *solutions of frequency* $\lambda$ *correspond to critical points of* $J_\lambda$ *on* $H_0^1(\Omega)$.

**Proposition**

*Given* $2 < p < 2^*$ *and* $\mu > 0$, *normalized solutions of mass* $\mu$ *correspond to constrained critical points of* $E$ *on the* $L^2$*-sphere*

$$\mathcal{M}_\mu := \left\{ u \in H_0^1(\Omega) \mid \|u\|_{L^2(\Omega)} = \mu \right\}.$$

In the case of normalized solutions, the parameter $\lambda$ in the PDE will appear as a Lagrange multiplier associated with the constraint.

# Lower boundedness of the energy functional

## Proposition

*Let* $2 < p < 2^*$ *and* $\mu > 0$. *Then:*

- *if* $2 < p < 2 + 4/N$,

$$\inf_{\mathcal{M}_\mu} E > -\infty;$$

- *if* $2 + 4/N < p < 2^*$,

$$\inf_{\mathcal{M}_\mu} E = -\infty.$$

# Lower boundedness of the energy functional

## Proposition

*Let* $2 < p < 2^*$ *and* $\mu > 0$. *Then:*

- *if* $2 < p < 2 + 4/N$,
$$\inf_{\mathcal{M}_\mu} E > -\infty;$$
- *if* $2 + 4/N < p < 2^*$,
$$\inf_{\mathcal{M}_\mu} E = -\infty.$$

The boundedness follows from the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p} \leq C(p) \|u\|_{L^2}^{1-s} \|\nabla u\|_{L^2}^{s}, \quad s := \frac{(p-2)N}{2p}.$$

# Lower boundedness of the energy functional

## Proposition

*Let* $2 < p < 2^*$ *and* $\mu > 0$. *Then:*

- *if* $2 < p < 2 + 4/N$,

$$\inf_{\mathcal{M}_\mu} E > -\infty;$$

- *if* $2 + 4/N < p < 2^*$,

$$\inf_{\mathcal{M}_\mu} E = -\infty.$$

The boundedness follows from the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p} \leq C(p)\,\|u\|_{L^2}^{1-s}\ \|\nabla u\|_{L^2}^{s}, \quad s := \frac{(p-2)N}{2p}.$$

and the unboundedness by considering the limit $t \to +\infty$ for a family $t^{N/2}\psi(tx)$, with constant $L^2$-norms, obtained by scaling a fixed profile.

# A classic result and two questions

## Proposition

*When* $\mu > 0$ *and* $2 < p < 2 + 4/N$, *then minimizers for* $E$ *on* $\mathcal{M}_\mu$ *exist, have a constant sign and are normalized solutions of* (NLS). *They are called energy ground states.*

# A classic result and two questions

## Proposition

*When $\mu > 0$ and $2 < p < 2 + 4/N$, then minimizers for $E$ on $\mathcal{M}_\mu$ exist, have a constant sign and are normalized solutions of $(\mathrm{NLS})$. They are called energy ground states.*

## Question

*Given $\mu > 0$ and $2 + 4/N < p < 2^*$, do there exist normalized solutions of mass $\mu$? Is there a least energy normalized solution?*

# A classic result and two questions

## Proposition

*When $\mu > 0$ and $2 < p < 2 + 4/N$, then minimizers for $E$ on $\mathcal{M}_\mu$ exist, have a constant sign and are normalized solutions of $(\mathrm{NLS})$. They are called energy ground states.*

## Question

*Given $\mu > 0$ and $2 + 4/N < p < 2^*$, do there exist normalized solutions of mass $\mu$? Is there a least energy normalized solution?*

## Question

*How to find sign-changing normalized solutions?*

# A classic result and two questions

## Proposition

*When $\mu > 0$ and $2 < p < 2 + 4/N$, then minimizers for $E$ on $\mathcal{M}_\mu$ exist, have a constant sign and are normalized solutions of (NLS). They are called energy ground states.*

## Question

*Given $\mu > 0$ and $2 + 4/N < p < 2^*$, do there exist normalized solutions of mass $\mu$? Is there a least energy normalized solution?*

## Question

*How to find sign-changing normalized solutions?*

Answers: given by the results of the talk!

# The fixed frequency case

In the fixed frequency case, we are a priori looking for critical points of an unconstrained functional.

# The fixed frequency case

In the fixed frequency case, we are a priori looking for critical points of an unconstrained functional.

However, the functional $J_\lambda$ is not bounded from below on $H_0^1(\Omega)$, since if $u \neq 0$ then

$$J_\lambda(tu) = \frac{t^2}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{\lambda t^2}{2}\|u\|_{L^2(\Omega)}^2 - \frac{t^p}{p}\|u\|_{L^p(\Omega)}^p \xrightarrow[t\to+\infty]{} -\infty.$$

# The Nehari manifold

A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda$, defined by

$$\begin{aligned}\mathcal{N}_\lambda &:= \Big\{ u \in H_0^1(\Omega) \setminus \{0\} \mid J_\lambda'(u)[u] = 0 \Big\} \\ &= \Big\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p \Big\}.\end{aligned}$$

# The Nehari manifold

A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda$, defined by

$$\begin{aligned}\mathcal{N}_\lambda &:= \left\{u \in H^1_0(\Omega) \setminus \{0\} \mid J'_\lambda(u)[u] = 0\right\} \\ &= \left\{u \in H^1_0(\Omega) \setminus \{0\} \mid \|\nabla u\|^2_{L^2(\Omega)} + \lambda \|u\|^2_{L^2(\Omega)} = \|u\|^p_{L^p(\Omega)}\right\}.\end{aligned}$$

If $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^p_{L^p(\Omega)}.$$

In particular, $J_\lambda$ is bounded from below on $\mathcal{N}_\lambda$.

# The Nehari manifold

A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda$, defined by

$$\begin{aligned}\mathcal{N}_\lambda &:= \left\{u \in H_0^1(\Omega) \setminus \{0\} \mid J'_\lambda(u)[u] = 0\right\} \\ &= \left\{u \in H_0^1(\Omega) \setminus \{0\} \mid \|\nabla u\|^2_{L^2(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)} = \|u\|^p_{L^p(\Omega)}\right\}.\end{aligned}$$

If $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^p_{L^p(\Omega)}.$$

In particular, $J_\lambda$ is bounded from below on $\mathcal{N}_\lambda$.

**Proposition**

*Given $\lambda > -\lambda_1(\Omega)$ and $2 < p < 2^*$, then minimizers for $J_\lambda$ on $\mathcal{N}_\lambda$ exist, have a constant sign and are solutions of $(\mathrm{NLS})$ having frequency $\lambda$. They are called action ground states.*

# Nodal action ground states

One defines the nodal Nehari set by

$$\mathcal{N}_\lambda^{nod} := \left\{ u \in H_0^1(\Omega) \mid u^\pm \in \mathcal{N}_\lambda(\Omega) \right\}.$$

# Nodal action ground states

One defines the nodal Nehari set by

$$\mathcal{N}_\lambda^{nod} := \left\{ u \in H_0^1(\Omega) \mid u^\pm \in \mathcal{N}_\lambda(\Omega) \right\}.$$

It contains all sign-changing solutions of $(\mathrm{NLS})$.

# Nodal action ground states

One defines the nodal Nehari set by

$$\mathcal{N}_\lambda^{nod} := \left\{ u \in H_0^1(\Omega) \mid u^\pm \in \mathcal{N}_\lambda(\Omega) \right\}.$$

It contains all sign-changing solutions of $(\mathrm{NLS})$.

**Theorem (Castro, Cossio, Neuberger 1997; Bartsch-Weth 2003)**

*Given $\lambda > -\lambda_2(\Omega)$ and $2 < p < 2^*$, then minimizers for $J_\lambda$ on $\mathcal{N}_\lambda^{nod}$ exist, have two nodal zones and are solutions of $(\mathrm{NLS})$ having frequency $\lambda$. They are called nodal action ground states.*

# Nodal action ground states

One defines the nodal Nehari set by

$$\mathcal{N}_\lambda^{nod} := \left\{ u \in H_0^1(\Omega) \mid u^\pm \in \mathcal{N}_\lambda(\Omega) \right\}.$$

It contains all sign-changing solutions of $(\mathrm{NLS})$.

**Theorem (Castro, Cossio, Neuberger 1997; Bartsch-Weth 2003)**

*Given $\lambda > -\lambda_2(\Omega)$ and $2 < p < 2^*$, then minimizers for $J_\lambda$ on $\mathcal{N}_\lambda^{nod}$ exist, have two nodal zones and are solutions of $(\mathrm{NLS})$ having frequency $\lambda$. They are called nodal action ground states.*

*Remark: I will use the terms "sign-changing" and "nodal" interchangeably, as the contrary of "one-signed".*

# Comparison of the two settings so far

Abbreviation: "ground state" $\rightarrow$ GS

| | $2 < p < 2 + 4/N$ | $2 + 4/N < p < 2^*$ |
|---|---|---|
| Positive solution | Energy GS | ? |
| Sign-changing solution | ? | ? |

The fixed mass $\mu$ case

# Comparison of the two settings so far

Abbreviation: "ground state" $\to$ GS

| | $2 < p < 2 + 4/N$ | $2 + 4/N < p < 2^*$ |
|---|---|---|
| Positive solution | Energy GS | ? |
| Sign-changing solution | ? | ? |

The fixed mass $\mu$ case

| | $2 < p < 2 + 4/N$ | $2 + 4/N < p < 2^*$ |
|---|---|---|
| Positive solution | Action GS | Action GS |
| Sign-changing solution | Nodal action GS | Nodal action GS |

The fixed action $\lambda$ case

# Sign-changing normalized solutions

The only work I am aware of which studies sign-changing normalized solutions is a recent preprint of Jeanjean and Song in 2025, using gradient flow techniques.

# Sign-changing normalized solutions

The only work I am aware of which studies sign-changing normalized solutions is a recent preprint of Jeanjean and Song in 2025, using gradient flow techniques.

In the literature, there is no equivalent of the nodal Nehari set for normalized solutions and it is in fact very unclear if such a nice "codimension two constraint" does exist for this problem.

# Positive normalized solutions in the $L^2$-supercritical regime

Since pioneering work by Jeanjean in the late 90s, there have been many studies devoted to the existence of positive normalized normalized solutions in the $L^2$-supercritical regime $2 + 4/N < p < 2^*$.

# Positive normalized solutions in the $L^2$-supercritical regime

Since pioneering work by Jeanjean in the late 90s, there have been many studies devoted to the existence of positive normalized normalized solutions in the $L^2$-supercritical regime $2 + 4/N < p < 2^*$.

Two main difficulties have to be taken into account:

- the energy is unbounded from below on the constraint $\rightarrow$ one obtains "mountain pass" solutions;

# Positive normalized solutions in the $L^2$-supercritical regime

Since pioneering work by Jeanjean in the late 90s, there have been many studies devoted to the existence of positive normalized normalized solutions in the $L^2$-supercritical regime $2 + 4/N < p < 2^*$.

Two main difficulties have to be taken into account:

- the energy is unbounded from below on the constraint $\rightarrow$ one obtains "mountain pass" solutions;
- it is unclear that Palais-Smale sequences associated to the variational problem are bounded!!!

# Positive normalized solutions in the $L^2$-supercritical regime

Since pioneering work by Jeanjean in the late 90s, there have been many studies devoted to the existence of positive normalized normalized solutions in the $L^2$-supercritical regime $2 + 4/N < p < 2^*$.

Two main difficulties have to be taken into account:

- the energy is unbounded from below on the constraint $\rightarrow$ one obtains "mountain pass" solutions;
- it is unclear that Palais-Smale sequences associated to the variational problem are bounded!!!

To resolve the second issue, one either uses the Pohožaev identity or a monotonicty trick "à la Struwe", later improved by Jeanjean and coauthors.

# Positive normalized solutions in the $L^2$-supercritical regime

Since pioneering work by Jeanjean in the late 90s, there have been many studies devoted to the existence of positive normalized normalized solutions in the $L^2$-supercritical regime $2 + 4/N < p < 2^*$.

Two main difficulties have to be taken into account:

- the energy is unbounded from below on the constraint $\rightarrow$ one obtains "mountain pass" solutions;
- it is unclear that Palais-Smale sequences associated to the variational problem are bounded!!!

To resolve the second issue, one either uses the Pohožaev identity or a monotonicty trick "à la Struwe", later improved by Jeanjean and coauthors.

While remarkably successful for autonomous PDEs set on $\mathbb{R}^N$, those techniques impose a lot of restrictions on the domain under study.

# The $L^2$-supercritical regime on domains

There a few works about positive normalized solutions on bounded domains (though significantly less than on $\mathbb{R}^N$):

# The $L^2$-supercritical regime on domains

There a few works about positive normalized solutions on bounded domains (though significantly less than on $\mathbb{R}^N$):

- Noris-Tavares-Verzini 2014 studied this problem *on the ball*, relying heavily on the uniqueness of the positive solution for a given $\lambda$ (based on seminal results of Gidas-Ni-Nirenberg and Kwong). Their analysis is very precise, but limited to a specific domain;

# The $L^2$-supercritical regime on domains

There a few works about positive normalized solutions on bounded domains (though significantly less than on $\mathbb{R}^N$):

- Noris-Tavares-Verzini 2014 studied this problem *on the ball*, relying heavily on the uniqueness of the positive solution for a given $\lambda$ (based on seminal results of Gidas-Ni-Nirenberg and Kwong). Their analysis is very precise, but limited to a specific domain;
- Pierotti-Verzini 2016 then Pierotti-Verzini-Yu 2025 considered general bounded domains. Good existence results require star-shapedness of the domain, a quite strong geometrical assumption.

# The $L^2$-supercritical regime on domains

There a few works about positive normalized solutions on bounded domains (though significantly less than on $\mathbb{R}^N$):

- Noris-Tavares-Verzini 2014 studied this problem *on the ball*, relying heavily on the uniqueness of the positive solution for a given $\lambda$ (based on seminal results of Gidas-Ni-Nirenberg and Kwong). Their analysis is very precise, but limited to a specific domain;
- Pierotti-Verzini 2016 then Pierotti-Verzini-Yu 2025 considered general bounded domains. Good existence results require star-shapedness of the domain, a quite strong geometrical assumption.

  Notably, the authors point out that, in the $L^2$-supercritical regime on a bounded domain, sequences of solutions having a bounded Morse index are bounded in $L^2$.

# Action versus energy ground states

While both notions of action GS and of energy GS have a long history, most papers studied either one or the other, and a comparison of both notions was not considered until rather recently.

# Action versus energy ground states

While both notions of action GS and of energy GS have a long history, most papers studied either one or the other, and a comparison of both notions was not considered until rather recently.

**Proposition (Dovetta-Serra-Tilli(*) 2022)**

*Let $2 < p < 2 + 4/N$ and $\Omega$ be bounded.*
*Then if energy ground states do exist, they are necessarily action ground states for the corresponding $\lambda$. The converse is not necessarily true!*

# Action versus energy ground states

While both notions of action GS and of energy GS have a long history, most papers studied either one or the other, and a comparison of both notions was not considered until rather recently.

**Proposition (Dovetta-Serra-Tilli(*) 2022)**

*Let* $2 < p < 2 + 4/N$ *and* $\Omega$ *be bounded.*
*Then if energy ground states do exist, they are necessarily action ground states for the corresponding* $\lambda$*. The converse is not necessarily true!*

(*) This statement was more or less known in the literature before the DST paper, but not considered from the point of view of the systematic comparison of both notions of GS.

# Action versus energy ground states (continued)

## Theorem (Dovetta-Serra-Tilli 2022)

*Let* $2 < p < 2 + 4/N$ *and* $\Omega$ *be bounded.*

*For any* $\mu > 0$, *define*

$$\mathcal{E}(\mu) := \inf_{u \in \mathcal{M}_\mu} E(u)$$

*and, for every* $\lambda \in \mathbb{R}$, *define*

$$\mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u).$$

# Action versus energy ground states (continued)

## Theorem (Dovetta-Serra-Tilli 2022)

*Let* $2 < p < 2 + 4/N$ *and* $\Omega$ *be bounded.*

*For any* $\mu > 0$*, define*

$$\mathcal{E}(\mu) := \inf_{u \in \mathcal{M}_\mu} E(u)$$

*and, for every* $\lambda \in \mathbb{R}$*, define*

$$\mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u).$$

*Then,* $-\mathcal{E}(2\mu)$ *is the Legendre-Fenchel transform of* $\mathcal{J}$*. Namely, one has*

$$-\mathcal{E}(2\mu) = \sup_{\lambda \in \mathbb{R}} \Big(\lambda\mu - \mathcal{J}(\lambda)\Big).$$

# Main message

In their paper, Dovetta, Serra and Tilli *compare two families of solutions whose existence is known a priori via minimization procedures: the action GS and the energy GS.*

# Main message

In their paper, Dovetta, Serra and Tilli *compare two families of solutions whose existence is known a priori via minimization procedures: the action GS and the energy GS.*

## Main message

The convex duality we just saw is a *method* !!!

# Main message

In their paper, Dovetta, Serra and Tilli *compare two families of solutions whose existence is known a priori via minimization procedures: the action GS and the energy GS.*

## Main message

The convex duality we just saw is a *method* !!!

More precisely:

- using such a "convex duality argument" from the action ground states when $2 + 4/N < p < 2^*$ will *also* produce normalized solutions;

# Main message

In their paper, Dovetta, Serra and Tilli *compare two families of solutions whose existence is known a priori via minimization procedures: the action GS and the energy GS.*

## Main message

The convex duality we just saw is a *method* !!!

More precisely:

- using such a "convex duality argument" from the action ground states when $2 + 4/N < p < 2^*$ will *also* produce normalized solutions;
- doing so from the nodal action GS will produce sign-changing normalized solutions, which is new for all $2 < p < 2^*$.

# Our result (for positive solutions)

**Theorem (De Coster-Dovetta-G.-Serra 2025)**

*Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $2 < p < 2^*$, let*

$$M_p := \left\{ \|u\|^2_{L^2(\Omega)} \;\middle|\; u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\}$$

*be the set of masses of all action ground states. Then,*

# Our result (for positive solutions)

## Theorem (De Coster-Dovetta-G.-Serra 2025)

*Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $2 < p < 2^*$, let*

$$M_p := \left\{ \|u\|^2_{L^2(\Omega)} \mid u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\}$$

*be the set of masses of all action ground states. Then,*

(i) *if $2 < p < 2 + 4/N$, then $M_p(\Omega) = (0, +\infty)$;*

# Our result (for positive solutions)

## Theorem (De Coster-Dovetta-G.-Serra 2025)

*Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $2 < p < 2^*$, let*

$$M_p := \left\{ \|u\|^2_{L^2(\Omega)} \;\middle|\; u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\}$$

*be the set of masses of all action ground states. Then,*

(i) *if $2 < p < 2 + 4/N$, then $M_p(\Omega) = (0, +\infty)$;*

(ii) *if $2 + 4/N < p < 2^*$, then there exist $0 < \mu_p < +\infty$ such that $M_p = (0, \mu_p]$.*

# *Isn't that quite obvious?*

One may argue that obtaining *intervals of masses* is a trivial consequence of the intermediate value theorem.

# *Isn’t that quite obvious?*

One may argue that obtaining *intervals of masses* is a trivial consequence of the intermediate value theorem.

This would be true *if* the map $\lambda \mapsto u_\lambda$ mapping $\lambda$ to the action GS had good continuity properties, *which is expected to be wrong in general!*

# *Isn't that quite obvious?*

One may argue that obtaining *intervals of masses* is a trivial consequence of the intermediate value theorem.

This would be true *if* the map $\lambda \mapsto u_\lambda$ mapping $\lambda$ to the action GS had good continuity properties, *which is expected to be wrong in general!*

In fact, this map is not even well-defined as action GS might not be unique.

# Properties of the action level map $\lambda \mapsto \mathcal{J}(\lambda)$

## Proposition

*Let* $2 < p < 2^*$. *Then:*

# Properties of the action level map $\lambda \mapsto \mathcal{J}(\lambda)$

## Proposition

*Let* $2 < p < 2^*$. *Then:*

(i) *For every* $\lambda \leq -\lambda_1$, $\mathcal{J}(\lambda) = 0$ *and action ground states in* $\mathcal{N}_\lambda(\Omega)$ *do not exist.*

# Properties of the action level map $\lambda \mapsto \mathcal{J}(\lambda)$

## Proposition

*Let $2 < p < 2^*$. Then:*

(i) *For every $\lambda \leq -\lambda_1$, $\mathcal{J}(\lambda) = 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ do not exist.*

(ii) *For every $\lambda > -\lambda_1$, $\mathcal{J}(\lambda) > 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ exist.*

# Properties of the action level map $\lambda \mapsto \mathcal{J}(\lambda)$

## Proposition

*Let $2 < p < 2^*$. Then:*

(i) *For every $\lambda \leq -\lambda_1$, $\mathcal{J}(\lambda) = 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ do not exist.*

(ii) *For every $\lambda > -\lambda_1$, $\mathcal{J}(\lambda) > 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ exist.*

(iii) *The function $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\lambda_1, +\infty)$.*

# Properties of the action level map $\lambda \mapsto \mathcal{J}(\lambda)$

## Proposition

*Let $2 < p < 2^*$. Then:*

(i) *For every $\lambda \leq -\lambda_1$, $\mathcal{J}(\lambda) = 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ do not exist.*

(ii) *For every $\lambda > -\lambda_1$, $\mathcal{J}(\lambda) > 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ exist.*

(iii) *The function $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\lambda_1, +\infty)$.*

Moreover, "derivatives of $\mathcal{J}$ give $L^2$-masses of action ground states" (to be precised).

# A completely wrong argument

## But still a good heuristic :-)

Let $u \in H_0^1(\Omega)$ be **fixed**.

# A completely wrong argument

## But still a good heuristic :-)

Let $u \in H_0^1(\Omega)$ be **fixed**.

Recalling the definition of $J_\lambda$, we have

$$J_\lambda(u) = E(u) + \frac{\lambda}{2}\|u\|_{L^2}^2,$$

# A completely wrong argument

## But still a good heuristic :-)

Let $u \in H_0^1(\Omega)$ be **fixed**.

Recalling the definition of $J_\lambda$, we have

$$J_\lambda(u) = E(u) + \frac{\lambda}{2}\|u\|_{L^2}^2,$$

so that

$$\partial_\lambda\Big(J_\lambda(u)\Big) = \frac{1}{2}\|u\|_{L^2}^2.$$

# A completely wrong argument

## But still a good heuristic :-)

Let $u \in H_0^1(\Omega)$ be **fixed**.

Recalling the definition of $J_\lambda$, we have

$$J_\lambda(u) = E(u) + \frac{\lambda}{2}\|u\|_{L^2}^2,$$

so that

$$\partial_\lambda\Big(J_\lambda(u)\Big) = \frac{1}{2}\|u\|_{L^2}^2.$$

Of course, we have that $\mathcal{J}(\lambda) = J_\lambda(u_\lambda)$ for a **varying** action GS $u_\lambda$ (they must be in different Nehari manifolds!). It just so happens that the action GS change "little enough" that the leading term is the same than if the minimizer was fixed, which is extremely convenient.

# A correct version of the heuristic argument

## Proposition

*Let $2 < p < 2^*$ and define*

$$Q_p(\lambda) := \left\{ \|u\|_2^2 \mid u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u, \Omega) = \mathcal{J}(\lambda) \right\}.$$

*be the set of masses of action ground states.*

# A correct version of the heuristic argument

## Proposition

*Let* $2 < p < 2^*$ *and define*

$$Q_p(\lambda) := \left\{ \|u\|_2^2 \mid u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u,\Omega) = \mathcal{J}(\lambda) \right\}.$$

*be the set of masses of action ground states. Then, we have*

$$\begin{aligned}\lim_{\varepsilon \to 0^+} \frac{\mathcal{J}(\lambda+\varepsilon) - \mathcal{J}(\lambda)}{\varepsilon} &= \frac{1}{2} \inf Q_p(\lambda) \\ &\leq \frac{1}{2} \sup Q_p(\lambda) = \lim_{\varepsilon \to 0^-} \frac{\mathcal{J}(\lambda+\varepsilon) - \mathcal{J}(\lambda)}{\varepsilon},\end{aligned}$$

# A correct version of the heuristic argument

## Proposition

*Let* $2 < p < 2^*$ *and define*

$$Q_p(\lambda) := \left\{ \|u\|_2^2 \mid u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u,\Omega) = \mathcal{J}(\lambda) \right\}.$$

*be the set of masses of action ground states. Then, we have*

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{\mathcal{J}(\lambda+\varepsilon) - \mathcal{J}(\lambda)}{\varepsilon} &= \frac{1}{2} \inf Q_p(\lambda) \\ &\leq \frac{1}{2} \sup Q_p(\lambda) = \lim_{\varepsilon \to 0^-} \frac{\mathcal{J}(\lambda+\varepsilon) - \mathcal{J}(\lambda)}{\varepsilon}, \end{aligned}$$

*Moreover, for every* $\lambda$ *outside an at most countable set, all action ground states have the same mass (i.e.,* $Q_p(\lambda)$ *is a singleton).*

# A miracle

## Proposition (Key proposition)

*Let $\mu > 0$ and $2 < p < 2^*$. Assume that $\lambda_* > -\lambda_1(\Omega)$ is a local minima of the map $f_\mu : [-\lambda_1, +\infty) \to \mathbb{R}$ defined by*

$$f_\mu(\lambda) := \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda.$$

*Then, $\mathcal{J}$ is differentiable for $\lambda = \lambda_*$ and one has that $\mathcal{J}'(\lambda_*) = \mu$, so that all action ground states with $\lambda = \lambda_*$ have mass $\mu$.*

# Proof of the key proposition

## Proof.

At a minimum point, one must have

$$\limsup_{\varepsilon \to 0^-} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon} \leq 0 \leq \liminf_{\varepsilon \to 0^+} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon},$$

□

# Proof of the key proposition

## Proof.

At a minimum point, one must have

$$\limsup_{\varepsilon \to 0^-} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon} \leq 0 \leq \liminf_{\varepsilon \to 0^+} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon},$$

namely

$$\limsup_{\varepsilon \to 0^-} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon} \leq \liminf_{\varepsilon \to 0^+} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon}.$$

□

# Proof of the key proposition

## Proof.

At a minimum point, one must have

$$\limsup_{\varepsilon\to 0^-} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon} \leq 0 \leq \liminf_{\varepsilon\to 0^+} \frac{f_\mu(\lambda_* + \varepsilon) - f_\mu(\lambda_*)}{\varepsilon},$$

namely

$$\limsup_{\varepsilon\to 0^-} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon} \leq \liminf_{\varepsilon\to 0^+} \frac{\mathcal{J}(\lambda_* + \varepsilon) - \mathcal{J}(\lambda_*)}{\varepsilon}.$$

But we just saw that the reverse inequality holds! □

# Interlude: Darboux's Theorem for derivatives

Somehow, we just proved a “Darboux-type” result theorem for $\mathcal{J}'$ (even though $\mathcal{J}'$ is not pointwise well-defined). As a comparison, here is Darboux's original theorem.

**Theorem (Darboux 1875)**

*Let $f : I \to \mathbb{R}$ be differentiable, where $I$ is an interval. Then, $f'(I)$ is an interval.*

# Jean-Gaston Darboux (1842 – 1917)

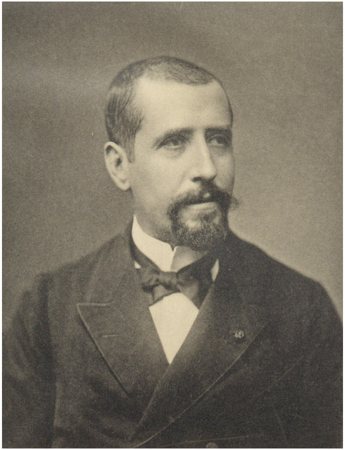

Image from Wikimedia Commons.

# Asymptotic behavior of $\mathcal{J}$: $\lambda \to -\lambda_1$

## Proposition

*For every* $2 < p < 2^*$, *there exist* $C_1, C_2 > 0$ *such that for every* $\lambda \geq -\lambda_1$,

$$\mathcal{J}(\lambda) \leq C_1(\lambda + \lambda_1)^{\frac{p}{p-2}}$$

$$\mathcal{J}(\lambda) \geq C_2 \min\left(1, \frac{\lambda + \lambda_1}{\lambda_1}\right)^{\frac{p}{p-2}}.$$

# Asymptotic behavior of $\mathcal{J}$: $\lambda \to -\lambda_1$

## Proposition

*For every* $2 < p < 2^*$, *there exist* $C_1, C_2 > 0$ *such that for every* $\lambda \geq -\lambda_1$,

$$\mathcal{J}(\lambda) \leq C_1(\lambda + \lambda_1)^{\frac{p}{p-2}}$$

$$\mathcal{J}(\lambda) \geq C_2 \min\left(1, \frac{\lambda + \lambda_1}{\lambda_1}\right)^{\frac{p}{p-2}}.$$

*In particular,*

$$\frac{\mathcal{J}(\lambda)}{\lambda + \lambda_1} \xrightarrow[\lambda \to -\lambda_1]{\lambda > -\lambda_1} 0.$$

# Asymptotic behavior of $\mathcal{J}$: $\lambda \to +\infty$

## Proposition

*We have*

$$\lim_{\lambda \to +\infty} \frac{\mathcal{J}(\lambda)}{\lambda} = \begin{cases} +\infty & \text{if } 2 < p < 2 + 4/N, \\ 0 & \text{if } 2 + 4/N < p < 2^*. \end{cases}$$

# Putting it all together

Using the asymptotic results, we are able to show that the map $\lambda \mapsto \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda$ has local minima:

# Putting it all together

Using the asymptotic results, we are able to show that the map $\lambda \mapsto \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda$ has local minima:

- for all $0 < \mu$ if $2 < p < 2 + 4/N$, in this case *one can even find global minima*;

# Putting it all together

Using the asymptotic results, we are able to show that the map $\lambda \mapsto \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda$ has local minima:

- for all $0 < \mu$ if $2 < p < 2 + 4/N$, in this case *one can even find global minima*;
- for all $0 < \mu < \overline{\mu}$ if $2 + 4/N < p < 2^*$, in which case the map does not have global minima (which is somehow a trace that we are dealing with the harder case where the energy is unbounded from below).

# Putting it all together

Using the asymptotic results, we are able to show that the map $\lambda \mapsto \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda$ has local minima:

- for all $0 < \mu$ if $2 < p < 2 + 4/N$, in this case *one can even find global minima*;
- for all $0 < \mu < \overline{\mu}$ if $2 + 4/N < p < 2^*$, in which case the map does not have global minima (which is somehow a trace that we are dealing with the harder case where the energy is unbounded from below).

This proves our announced results for positive solutions.

# What about sign-changing solutions?

Main comments:

# What about sign-changing solutions?

Main comments:

- $-\lambda_2(\Omega)$ now becomes the "natural threshold" in $\lambda$;

# What about sign-changing solutions?

Main comments:

- $-\lambda_2(\Omega)$ now becomes the "natural threshold" in $\lambda$;
- life gets harder when $\lambda \leq -\lambda_1(\Omega)$, essentially because the quadratic form

$$u \mapsto \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \lambda \int_\Omega |u|^2 \,\mathrm{d}x$$

ceases to be a norm;

# What about sign-changing solutions?

Main comments:

- $-\lambda_2(\Omega)$ now becomes the "natural threshold" in $\lambda$;
- life gets harder when $\lambda \leq -\lambda_1(\Omega)$, essentially because the quadratic form
$$u \mapsto \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \lambda \int_\Omega |u|^2 \, \mathrm{d}x$$
ceases to be a norm;
- we have to rely on Bartsch-Weth's (non-trivial!) result to obtain existence of nodal action ground states when $-\lambda_2 < \lambda \leq -\lambda_1$;

# What about sign-changing solutions?

Main comments:

- $-\lambda_2(\Omega)$ now becomes the "natural threshold" in $\lambda$;
- life gets harder when $\lambda \leq -\lambda_1(\Omega)$, essentially because the quadratic form
$$u \mapsto \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \lambda \int_\Omega |u|^2 \, \mathrm{d}x$$
ceases to be a norm;
- we have to rely on Bartsch-Weth's (non-trivial!) result to obtain existence of nodal action ground states when $-\lambda_2 < \lambda \leq -\lambda_1$;
- the claims can be adapted quite naturally to the nodal setting and proved in analogous ways, up to the above remarks. I refer to the paper for details!

# What are we looking for?

When $2 + 4/N < p < 2^*$, we saw that the energy functional is unbounded from below on the mass constraint.

# What are we looking for?

When $2 + 4/N < p < 2^*$, we saw that the energy functional is unbounded from below on the mass constraint.

We may however be interested in least energy normalized (nodal) solutions, namely solutions having least energy among all (nodal) solutions.

# What are we looking for?

When $2 + 4/N < p < 2^*$, we saw that the energy functional is unbounded from below on the mass constraint.

We may however be interested in least energy normalized (nodal) solutions, namely solutions having least energy among all (nodal) solutions.

For instance, Jeanjean's seminal 1997 paper produces least energy normalized solutions on $\mathbb{R}^N$.

# Pohožaev's identity

The following identity is often useful in the study of semilinear elliptic PDEs and follows by *clever* integration by parts.

## Proposition (Pohožaev's identity, 1965)

*Let* $2 < p < 2^*$, $\Omega$ *have a smooth boundary and* $u$ *be a solution to* $(\mathrm{NLS})$. *Then, one has*

$$\frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N}{p}\|u\|_p^p + \frac{\lambda N}{2}\|u\|_2^2 + \frac{1}{2}\int_{\partial\Omega} |\partial_\nu u|^2 x \cdot \nu \, d\sigma = 0.$$

# Pohožaev's identity

The following identity is often useful in the study of semilinear elliptic PDEs and follows by *clever* integration by parts.

**Proposition (Pohožaev's identity, 1965)**

*Let $2 < p < 2^*$, $\Omega$ have a smooth boundary and $u$ be a solution to* $(\mathrm{NLS})$. *Then, one has*

$$\frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N}{p}\|u\|_p^p + \frac{\lambda N}{2}\|u\|_2^2 + \frac{1}{2}\int_{\partial\Omega} |\partial_\nu u|^2 x \cdot \nu \, d\sigma = 0.$$

Remark: when $\Omega = \mathbb{R}^N$, *there is no boundary term!* This is why this identity is much more powerful on $\mathbb{R}^N$ than on domains.

# Star-shaped domains

## Corollary

*If $\Omega$ is star-shaped, then*

$$\frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N}{p}\|u\|_p^p + \frac{\lambda N}{2}\|u\|_2^2 \leq 0.$$

## Corollary

*If $\Omega$ is star-shaped and $u$ is a solution of* $(\mathrm{NLS})$, *then*

$$E(u) \geq \frac{N(p-p_c)}{4p}\|u\|_p^p, \quad p_c := 2 + 4/N.$$

*In particular, on star-shaped domains, all solutions have a positive energy in the $L^2$-supercritical case!*

# The result (for positive solutions)

**Theorem (De Coster-Dovetta-G.-Serra 2025)**

*Let $\Omega$ be bounded, open, smooth and star-shaped and $2 < p < 2^*$. Then:*

- *if $2 < p < 2 + 4/N$, then least energy normalized (nodal) solutions do exist for all masses;*
- *if $2 + 4/N < p < 2^*$, then least energy normalized (nodal) solutions do exist for all small masses.*

# The result (for positive solutions)

**Theorem (De Coster-Dovetta-G.-Serra 2025)**

*Let $\Omega$ be bounded, open, smooth and star-shaped and $2 < p < 2^*$. Then:*

- *if $2 < p < 2 + 4/N$, then least energy normalized (nodal) solutions do exist for all masses;*
- *if $2 + 4/N < p < 2^*$, then least energy normalized (nodal) solutions do exist for all small masses.*

Main idea: using the consequences of Pohožaev's identity, we show that solutions having a small mass must correspond to $\lambda$ close enough to $-\lambda_1$ (for GS) or to $-\lambda_2$ (for nodal GS), corresponding to cases we can handle with the "action approach".

# A counterintuitive fact when $p = 2 + 4/N$

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where $\mu_N$ is the mass of the corresponding soliton on $\mathbb{R}^N$.

# A counterintuitive fact when $p = 2 + 4/N$

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where $\mu_N$ is the mass of the corresponding soliton on $\mathbb{R}^N$.

Moreover, the analysis of Noris-Tavares-Verzini on the ball implies that the masses of all positive solutions are given *exactly* by $(0, \mu_N)$.

# A counterintuitive fact when $p = 2 + 4/N$

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where $\mu_N$ is the mass of the corresponding soliton on $\mathbb{R}^N$.

Moreover, the analysis of Noris-Tavares-Verzini on the ball implies that the masses of all positive solutions are given *exactly* by $(0, \mu_N)$.

Thus, on the ball, for $\mu \in [\mu_N, 2\mu_N)$, least energy nodal solutions exist, and there are no positive solutions, so that...

# A counterintuitive fact when $p = 2 + 4/N$

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where $\mu_N$ is the mass of the corresponding soliton on $\mathbb{R}^N$.

Moreover, the analysis of Noris-Tavares-Verzini on the ball implies that the masses of all positive solutions are given *exactly* by $(0, \mu_N)$.

Thus, on the ball, for $\mu \in [\mu_N, 2\mu_N)$, least energy nodal solutions exist, and there are no positive solutions, so that...

**In the critical and supercritical cases...**

least energy solutions may exist **and be nodal!**

# A counterintuitive fact when $p = 2 + 4/N$

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where $\mu_N$ is the mass of the corresponding soliton on $\mathbb{R}^N$.

Moreover, the analysis of Noris-Tavares-Verzini on the ball implies that the masses of all positive solutions are given *exactly* by $(0, \mu_N)$.

Thus, on the ball, for $\mu \in [\mu_N, 2\mu_N)$, least energy nodal solutions exist, and there are no positive solutions, so that...

**In the critical and supercritical cases...**

least energy solutions may exist **and be nodal!**

This strikingly shows that not all properties of energy ground states transfer to least energy normalized solutions.

# A (difficult?) open question

If $\Omega$ is not star-shaped, it is known that negative energy solutions can exist. This can be explored by studying such problems on metric graphs, which often lead to "simple" non-star-shaped domains.

# A (difficult?) open question

If $\Omega$ is not star-shaped, it is known that negative energy solutions can exist. This can be explored by studying such problems on metric graphs, which often lead to "simple" non-star-shaped domains.

**Question**

*Is there an intricate smooth bounded domain $\Omega$, an exponent $2 + 4/N < p < 2^*$ and a mass $\mu$ for which there exist a sequence of normalized solutions of mass $\mu$ whose energy go to $-\infty$?*

# A (difficult?) open question

If $\Omega$ is not star-shaped, it is known that negative energy solutions can exist. This can be explored by studying such problems on metric graphs, which often lead to "simple" non-star-shaped domains.

**Question**

*Is there an intricate smooth bounded domain $\Omega$, an exponent $2 + 4/N < p < 2^*$ and a mass $\mu$ for which there exist a sequence of normalized solutions of mass $\mu$ whose energy go to $-\infty$?*

My guess... *maybe yes, actually?*

*Merci beaucoup!*

# References

## Sign-changing normalized solutions

De Coster C., Dovetta S., Galant D., Serra E.
*An action approach to nodal and least energy normalized solutions for nonlinear Schrödinger equations.* ArXiV preprint:
https://arxiv.org/abs/2411.10317
Accepted for publication in Annales de l'Institut Henri Poincaré C - Analyse non linéaire (2025).

Jeanjean L., Song L.
*Sign-changing prescribed mass solutions for $L^2$-supercritical NLS on compact metric graphs.* ArXiV preprint:
https://arxiv.org/abs/2501.14642 (2025).

# References

## The nodal Nehari set


Castro A., Cossio J., Neuberger J.M,
*A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math. **27**(4), 1041–1053 (1997).

Bartsch T., Weth T.,
*A note on additional properties of sign changing solutions to superlinear elliptic equations*, Top. Meth. Nonlin. Anal. **22**, 1–14 (2003).

Szulkin A., Weth T.,
*The method of Nehari manifold*, Handbook of Nonconvex Analysis and Applications, D.Y. Gao and D. Motreanu eds., International Press, Boston, 597–632 (2010).

# References

## The $L^2$-supercritical case: unbounded domains


Jeanjean L.,
*Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlin. Anal. **28**(10), 1633–1659 (1997).

Bartsch T., de Valeriola S.
*Normalized Solutions of Nonlinear Schrödinger Equations.* Archiv der Mathematik, 100, 75–83 (2013).

Chang X., Jeanjean L., Soave N.
*Normalized solutions of $L^2$–supercritical NLS equations on compact metric graphs*, Ann. Inst. H. Poincaré (C) An. Non Lin., (2022).

Borthwick J., Chang X., Jeanjean L., Soave N.,
*Normalized solutions of $L^2$–supercritical NLS equations on noncompact metric graphs with localized nonlinearities*, Nonlinearity **36** (2023), 3776–3795.

# References

## The $L^2$-supercritical case: bounded domains

Noris B., Tavares H., Verzini G., *Existence and orbital stability of the ground states with prescribed mass for the $L^2$–critical and supercritical NLS on bounded domains*, Anal. PDE **7**(8), 1807–1838 (2014).

Pierotti D., Verzini G., *Normalized bound states for the nonlinear Schrödinger equation in bounded domains*, Calc. Var. PDE **56**, art. n. 133 (2017).

Pierotti D., Verzini G., Yu J., *Normalized solutions for Sobolev critical Schrödinger equations on bounded domains*, arXiv preprint 2404.04594 (2024).

# References

## Action versus energy

Jeanjean L., Lu S.-S,
*On global minimizers for a mass constrained problem*, Calc. Var. PDE **61**(6), art. n. 214 (2022).

Dovetta S., Serra E., Tilli P.,
*Action versus energy ground states in nonlinear Schrödinger equations*, Math. Ann. **385**, 1545–1576 (2023).

# References

## Pohožaev's identity


Pokhozhaev S.I.
*On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$.
Dokl. Akad. Nauk SSSR. 165: 36–39 (1965).

Struwe M.,
Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer Berlin, Heidelberg, fourth edition (2008).